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Gaussian Estimation of Single-Factor Continuous-Time Models of the South African Short-Term Interest Rate

A Dissertation presented to the School of Economics of the
University of Cape Town

in partial fulfilment of the requirements for the degree of
Master of Commerce

by

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Declaration

Hereby I, Peter David Aling, declare that this dissertation is my own original work and that all sources have been accurately reported and acknowledged, and that this document has not previously in its entirety or in part been submitted at any university in order to obtain an academic qualification.

P. D. Aling (ALNPET001)

12 February 2007

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Abstract

This paper presents the results of Gaussian estimation of the South African short-term interest rate. It uses the same Gaussian estimation techniques employed by Nowman (1997) to estimate the South African short-term interest rate using South african Treasury bill data. A range of single-factor continuous-time models of the short-term interest rate are estimated using a discrete-time model and compared to a discrete approximation used by Chan, Karolyi, Lonstaff and Sanders (1992a). We find that the process followed by the South African short-term interest rate is best explained by the Constant Elasticity of Variance (CEV) model and that the conditional volatility depends to some extent on the level of the interest rate. In addition we find evidence of a structural break in the mid-1980s, confirming our suspicions that the financial liberalisation of that period affected the short rate process.

Introduction

The short-term interest rate is a central input in the pricing of bonds, modelling the term structure of interest rates, and derivative security pricing models. Short-term interest rates are important in the development of tools for effective risk management and in many empirical studies analyzing term premiums and yield curves, where risk-free short-term interest rates are taken as reference rates for other interest rates. The short-term interest rate is therefore recognised as one of the most important prices determined in financial markets. As a result, there have been a plethora of models proposed to explain the evolution of the short-term interest rate.

Recent developments in econometric methods have seen a number of studies testing these models with actual interest rate data. These studies include Brown and Dybvig (1986), Melino and Turnbull (1986), Barone, Cuoco and Zautzik (1991), Babbs (1992), Abken (1993), Chen and Scott (1993), Das (1993), Gibbons and Ramaswamy (1993), Pearson and Sun (1994), Lund (1994), Pfann, Schotman and Tschernig (1995), Aït-Sahalia (1995), and Broze, Scaillet and Zakoian (1995). The seminal paper by Chan, Karolyi, Longstaff and Sanders (1992, hereafter CKLS) used the generalised method of moments (GMM) to estimate and compare a number of single-factor continuous-time models for the US short rate. They concluded that term structure models with volatilities that are more sensitive to the level of the risk-free interest rate perform better than more generally used models.

Following this study Nowman (1997) employed recently developed econometric techniques for the Gaussian estimation of continuous-time models (Bergstrom, 1983, 1985, 1986, 1990) to repeat the study for both US and UK data using both the discrete approximation employed by CKLS as well as an alternative discrete time model that nested the CKLS approximation but also had the advantage of reducing some of the temporal aggregation bias. He found that there was little difference between the two discrete time models. However, while the conclusions of CKLS held for US data, this

was not the case for UK data where the level of the interest rate had little effect on the volatility.

This study aims to replicate that of Nowman for the South African risk-free short-term interest rate. We estimate the models using the same Gaussian estimation techniques using South African Treasury bill (hereafter T-bill) data and find that the process followed by the South African short-term interest rate is best explained by the Constant Elasticity of Variance (CEV) model and that the conditional volatility depends to some extent on the level of the interest rate. In addition we find evidence of a structural break in the mid-1980s, confirming our suspicions that the financial liberalisation of that period affected the short rate process.

Section I below provides an overview of interest rate theory and has been included to keep the paper self-contained but can be skipped by readers already familiar with the theory or only interested in the empirical results. Section II reviews the single-factor continuous-time models used in CKLS. Section III describes the data and Gaussian estimation methodology developed by Bergstrom (1983, 1984, 1985, 1986, 1990) and used by Nowman (1997). Section IV describes the empirical results for the two estimation periods, 1957-2005 and 1985-2005, and Section V comprises a summary and conclusion.

Section I – A Review of Interest Rate Theory

The term structure of interest rates measures the relationship between yields on securities that differ only in their term to maturity. It is believed by many economists and investors that the term structure of interest rates conveys information about economic agents' expectations about future interest rates, inflation and exchange rates. An explanation of the term structure of interest rates therefore provides economists with the tools to extract that information and therefore predict how changes in underlying variables will affect the yield curve.

Much of the earlier literature focused on models of the term structure based on some version of the expectations hypothesis. Ingersoll (1987) outlines several common versions of the expectations hypothesis: the unbiased expectations hypothesis, the return-to-maturity expectations hypothesis, the yield-to-maturity expectations hypothesis and the local expectations hypothesis. All of these versions are shown to have severe weaknesses in explaining the term structure of interest rates. In the first three models the problem of unbounded interest rates arises. In addition, the resulting models of the term structure are not consistent across the different expectations models (Ingersoll, 1987).

More recent developments in interest rate theory have therefore focused on arbitrage theory in continuous-time to explain the term structure of interest rates. This theory is based on the fundamental economic assumption of the absence of arbitrage opportunities in the financial market considered. In other words, two portfolios having the same payoff at a given future date must have the same price today. This section will outline the no-arbitrage model of the term structure in continuous-time. Those readers interested in a more complete treatment of arbitrage theory in continuous-time are referred to Brigo and Mercurio (2001) and Björk (1998).

Portfolio Dynamics

We consider a market consisting of different assets such as stocks, bonds with various maturities, or various other financial derivatives. We begin by taking the price dynamics of these assets as given and proceed to derive the dynamics of (pricing of) a so-called self-financing portfolio (Björk, 1998, p.69).

Definition 1.1 (Björk, 1998, p.69, Definition 5.1) (Brigo and Mercurio, 2001, p.24, Definitions 2.1.1)

- 1.1.1 $n + 1$ = the number of different types of assets
- ϕ_i^t = the number of units of asset i held at time t
- ϕ_t = the portfolio $[\phi_t^0, \phi_t^1, \phi_t^2, \dots, \phi_t^n]$ held at time t
- S_i^t = the price of asset i at time t

1.1.2 A *portfolio* is a $(n + 1)$ dimensional process $\phi = \{\phi_t : 0 \leq t \leq T\}$, whose components $\phi^0, \phi^1, \phi^2, \dots, \phi^n$ are locally bounded and predictable.

1.1.3 The *value process* associated with a portfolio ϕ is defined by

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^n \phi_t^k S_t^k \quad 0 \leq t \leq T \quad (1)$$

1.1.4 A portfolio ϕ is *self-financing* if $V(\phi) \geq 0$ and

$$dV_t(\phi) = \phi_t dS_t = \sum_{i=0}^n \phi_t^i dS_t^i \quad 0 \leq t \leq T \quad (2)$$

Intuitively a portfolio's value at time t is equal to the number of assets held multiplied by their respective prices at time t . A portfolio is then defined as self-financing if there is no exogenous infusion or withdrawal of money (Björk, 2002, p.7). The purchase of new assets in the portfolio must be funded by the sales of assets already in the portfolio.

Arbitrage, Completeness and Martingales

Definition 1.2 (Brigo and Mercurio, 2001, p.25, Definition 2.1.2)

1.2.1. An admissible portfolio ϕ_t is called an *arbitrage* if the associated value process

$$V_t(\phi) \text{ satisfies } V_0(\phi) = 0 \text{ and } V_T(\phi) \geq 0 \text{ and } P[V_T(\phi) > 0] > 0$$

That is, a portfolio is said to be an arbitrage if it is self-financing and its value at time T is greater than 0 with positive probability. If no such portfolio exists then the economy is said to be arbitrage-free.

1.2.2. An *equivalent martingale measure* Q is a probability measure on the space (Ω, F) such that

(i) P and Q are equivalent measures, that is $P(A) = 0 \Leftrightarrow Q(A) = 0, \quad \forall A \in F$

(ii) The Radon-Nikodym derivative $\frac{dQ}{dP} \in L^2(\Omega, F, P)$

(iii) The "discounted asset price" process $e^{-\int_0^t r(s)ds} S$ is a Q -martingale, i.e.

$$\mathbb{E}^Q \left[e^{-\int_0^t r(s) ds} S_t^k | F_u \right] = e^{-\int_0^u r(s) ds} S_u^k \quad (3)$$

for all $k = 0, 1, \dots, n$ for all $0 \leq u \leq t \leq T$.

Informally, Q is an equivalent martingale measure if it assigns zero probabilities to all outcomes for which the probability is zero under the measure P . In addition, the Radon-Nicodym derivative, dQ/dP , must be integrable on the probability space Ω adapted to the filtration F under measure P . The filtration F is simply the information generated over the interval in question. The discounted asset price process must also be a Q -martingale. This leads us to the following theorem connecting the existence of an equivalent martingale measure and the absence of arbitrage.

Theorem 1.1 (Brigo and Mercurio, 2001, p.26)

Assume that there exists an equivalent martingale measure Q . Then the economy is free of arbitrage.

Definition 1.3 (Brigo and Mercurio, 2001, p.25, Definition 2.1.2)

- 1.3.1. A contingent T-claim is any random variable $X \in L^2(\Omega, F_T, P)$.
- 1.3.2. It is called attainable if $\exists \phi : V_T(\phi) = X$.
- 1.3.3. Such a ϕ is said to generate X , and $\pi_t = V_t(\phi)$ is the price at time t .
- 1.3.4. The economy is called complete if and only if every contingent claim is attainable.

That is, a contingent T-claim any random variable adapted to the filtration, F_T under measure P on the probability space Ω . It is attainable if there exists a portfolio with a value equal to X at time T and the price of such a portfolio at time t is π_t . If every contingent claim is attainable then the market is complete.

Theorem 1.2 (Brigo and Mercurio, 2001, p.26)

The economy is complete if and only if the martingale measure is unique.

Thus, the existence of a unique martingale measure makes the economy free of arbitrage and also allows the derivation of a unique price associated with any contingent claim. If the market is not complete then the price of any contingent claim will be different depending on the choice of Q .

Proposition 1.1 (Brigo and Mercurio, 2001, p.26, Proposition 2.1.2)

Assume there exists an equivalent martingale measure Q and let X be an attainable contingent claim. Then, for each time $t, 0 \leq t \leq T$, there exists a unique price π_t associated with X , i.e.,

$$\pi_t = E^Q \left[e^{-\int_t^T r(s) ds} X | F_t \right] \quad (4)$$

In summary, the market is arbitrage-free if there exists a martingale measure Q . If this measure is unique then the market is complete and every contingent claim can be uniquely priced. This brings us to the following metatheorem which allows us to easily determine whether a market is complete or arbitrage-free.

Metatheorem 1.3 (Björk, 2002, p.22, Metatheorem)

Assume that

N = Number of risky assets

R = Number of independent sources of randomness

Then the following hold

The market is arbitrage-free if $R \geq N$

The market is complete if $R \leq N$

The market is arbitrage-free and complete if $R = N$

Interest Rate Theory

We begin by considering a market with only one exogenously given (locally risk-free) asset. The price, B , of this asset has the following dynamics:

$$dB(t) = r(t)B(t)dt \quad (5)$$

where the dynamics of r , under the objective probability measure P are given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))d\bar{W} \quad (6)$$

and \bar{W} is a standard m -dimensional Wiener process.

It is clear that in the market we are considering there is one source of randomness $d\bar{W}$. However, there is no risky asset in this model. Therefore, from the metatheorem we can expect that the exogenously given market is arbitrage-free but not complete. The lack of completeness arises since we have no possibility of forming interesting portfolios: since the only exogenously given asset is the risk-free asset. This means that the price of any particular bond cannot be completely determined by the P -dynamics of r and the requirement that the market is free of arbitrage. This is because arbitrage pricing is always a case of pricing a derivative in terms of the price of some underlying assets. In this case we do not have sufficiently many underlying assets.

Fortunately though, the prices of bonds must satisfy certain internal consistency relations in order to avoid arbitrage possibilities on the bond market. If we take the price of some “benchmark” bond as given then prices of all the other bonds will be uniquely determined in terms of the price of the benchmark bond (and the r -dynamics) (Björk, 1998, p.245).

The Term Structure Equation (Björk, 2002, p.32-38)

We assume that there exists a market for T-bonds for every choice of T and that the market is arbitrage-free. Furthermore, the price of a T-bond has the form

$$\begin{aligned} p(t; T) &= F(t; r(t); T) \\ p(t; T) &= F^T(t; r(t)) \end{aligned}$$

At maturity the T-bond is worth 1 and thus

$$F^T(T; r) = 1 \text{ for all } r.$$

We form a portfolio of T and S bonds. We then apply Itô to $F^T(t, r(t))$ to get the following bond and portfolio dynamics,

$$dV = V \left\{ u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S} \right\}. \quad (7)$$

Solving for the portfolio weights and substituting back into equation (7) it can be shown that (Björk, 2002, p.35),

$$dV = V \left\{ \frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} \right\} dt. \quad (8)$$

Absence of arbitrage requires that

$$\frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} = r(t) \quad (9)$$

which can be written as

$$\frac{\alpha_s(t) - r(t)}{\sigma_s(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} \quad (10)$$

We note that the quotient is independent of the choice of maturity and thus we conclude that if the bond market is free of arbitrage then there exists some universal process $\lambda(t)$ such that the relation

$$\lambda(t) = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} \quad (11)$$

holds for all t and every for every choice of maturity, T . This process is known as the “market price of risk” or “risk premium per unit of volatility” (Björk, 1998, p.247).

We may obtain even more information from equation (11) by substituting in the earlier equations for α_T and σ_T . After some manipulation we obtain one of the most important equations in the theory of interest rates, the so called “term structure equation”.

Proposition 1.2 (The Term Structure Equation) (Björk, 1998, p.248, Proposition 16.2)

In an arbitrage-free bond market, F^T will satisfy the term structure equation

$$\begin{cases} F_t^T + \{\mu + \lambda\sigma\}F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F^T(T, r) = 1 \end{cases} \quad (12)$$

We obtain a Feynman-Kač representation of F^T by fixing (t, r) and then using the process

$$\exp\left\{-\int_t^s r(u)du\right\}F^T(s, r(s)). \quad (13)$$

If we apply the Itô formula to (13) and use the fact that F^T satisfies the term structure equation, then we can obtain the following stochastic representation formula:

Proposition 1.3 (Risk Neutral Valuation) (Björk, 1998, p.248, Proposition 16.3)

Bond prices are given by the formula $p(t, T) = F(t, r(t); T)$ where

$$F(t, r; T) = E_{t,r}^Q \left[e^{-\int_t^T r(s) ds} \right]. \quad (14)$$

Here the martingale measure Q and the subscripts t, r denote that the expectation shall be taken given the following dynamics for the short rate.

$$\begin{aligned} dr(s) &= \{\mu - \lambda\sigma\}ds + \sigma dW(s) \\ r(t) &= r. \end{aligned} \quad (15)$$

We see that the value of a T-bond at time t is given as the expected value of the final payoff of one discounted to the present value. The deflator used is the natural one, namely

$$e^{\left\{-\int_t^T r(s) ds\right\}}$$

but we observe that the expectation is not to be taken using the underlying objective probability measure P . Instead we must use the martingale measure Q and we see that we have different martingale measures for different choices of λ . (See Appendix A for a discussion on the change of numeraire and measure.)

This is due to the fact that the market is not complete and thus the various bond prices are determined only in part by the P -dynamics of the short rate and partly by other market forces.

The bonds treated above are, of course, contingent claims of a particularly simple type: they are deterministic. If we look at a more general type of contingent T-claim

$$\chi = \Phi(r(T)), \quad (16)$$

where Φ is some real-valued function. Using the same arguments as above we have the following result:

Proposition 1.4 (General Term Structure Equation) (Björk, 1998, p.249, Proposition 16.4)

Let χ be a contingent T-claim of the form $\chi = \Phi(r(T))$. In an arbitrage-free market the price $\Pi(t; \Phi)$ will be given as

$$\Pi(t; \Phi) = F(t, r(t)), \quad (17)$$

where F solves the boundary value problem

$$\begin{cases} F_t + \{\mu + \lambda\sigma\}F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0 \\ F(T, r) = \Phi(r) \end{cases} \quad (18)$$

Furthermore F has the stochastic representation

$$F(t, r; T) = E_{t,r}^Q \left[e^{-\int_t^T r(s)ds} \times \Phi(r(T)) \right], \quad (19)$$

where the martingale measure Q and the subscripts t, r denote that the expectation shall be taken using the following dynamics

$$\begin{aligned} dr(s) &= \{\mu - \lambda\sigma\}ds + \sigma dW(s) \\ r(t) &= r. \end{aligned} \quad (20)$$

The term structure (i.e. the complete family of bond price processes) can now be determined by the general term structure equation (18) as soon as we have specified the following objects:

- The drift term μ
- The diffusion term σ
- The market price of risk λ

Consider for a moment σ to be given *a priori*. Then it is clear from equation (18) that it is irrelevant exactly how we specify μ and λ *per se*. The object, apart from σ , that really determines the term structure (and all other derivatives) is the term $\mu - \lambda\sigma$ in equation (18). Now, from Proposition 1.4, we recall that the term $\mu - \lambda\sigma$ is the drift term of the short rate of interest under the martingale measure Q .

Therefore instead of specifying the μ and λ under the objective probability measure P we will instead specify the dynamics of the short rate r directly under martingale measure Q . This procedure is known as martingale modelling, and the typical assumption will be that r under Q has dynamics given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \quad (21)$$

where μ and σ are given functions. From now on, the letter μ will denote the drift term for the short rate of interest under the martingale measure Q .

In the literature there are a large number of proposals on how to specify the Q -dynamics for r . In the next section we examine a (far from complete) list of the popular single-factor models (Björk, 1998, p.252-253).

Section II – Short Rate Models

As a first step in modelling short-term interest rates, one-factor models of the term structure of interest rates form the basic building blocks for more complex models. Thus, finding an adequate characterization of the short-term interest rate will help determine if one-factor models of the term structure may be applied to South African interest rates.

CKLS chose the following general stochastic differential equation to specify the dynamic adjustment of the short interest rate:

$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^\gamma(t)dZ \quad (t \geq 0) \quad (22)$$

where $\{r(t), t \geq 0\}$ is a real continuous-time random process, and α, β, γ and σ are unknown structural parameters. Thus, both the drift, $\alpha + \beta r(t)$, and the conditional variance of the interest rate process, $\sigma^2 r^{2\gamma}(t)dt$, depend upon the level of the interest rate. However, instead of assuming that Z is a geometric Brownian motion process as in CKLS, we follow Bergstrom (1983, 1984 Theorem 2) as Nowman (1997) did and assume the following about dZ .

Assumption 1 (Nowman, 1997, p.1696, Assumption 1)

Z is a random measure defined on all subsets of the half line $0 < t < \infty$ with finite Lebesgue measure, such that $E[dZ] = 0$ and $E[dZ^2] = dt$ and $E[Z(\Delta_1)Z(\Delta_2)] = 0$ for any disjoint sets Δ_1 and Δ_2 on the half line $0 < t < \infty$.

This is weaker than the assumption that the innovations are generated by Brownian motion. The assumptions about the innovation process include the case where the innovations are a mixture of Brownian motion and Poisson processes and allow for more general innovation processes in which the increments are not independent but merely orthogonal (Nowman, 1997, p.1696).

Standard interest rate models can be obtained from equation (22) by imposing restrictions on the parameters α , β , γ and σ . The resulting specifications are listed below and the parameter restrictions are summarised in Table 1.

1. Merton (1973)	$dr(t) = \alpha dt + \sigma dZ$
2. Vasicek (1977)	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$
3. Cox, Ingersoll, and Ross (1985)	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$
4. Dothan (1978)	$dr(t) = \sigma r(t)dZ$
5. Geometric Brownian Motion	$dr(t) = \beta r(t)dt + \sigma r(t)dZ$
6. Brennan and Schwartz (1980)	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$
7. Cox, Ingersoll, and Ross (1980)	$dr(t) = \sigma r^{3/2}(t)dZ$
8. Constant Elasticity of Variance	$dr(t) = \beta r(t)dt + \sigma r^\gamma(t)dZ$

Table I
Parameter Restrictions Imposed by Alternative Models of Short-Term Interest Rate
The alternative term structure models for r are obtained by imposing the appropriate parameter restrictions on the unrestricted model

		$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^\gamma(t)dZ$			
	Model	α	β	σ^2	γ
Merton	$dr(t) = \alpha dt + \sigma dZ$		0		0
Vasicek	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$				0
CIR SR	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$				$1/2$
Dothan	$dr(t) = \sigma r(t)dZ$	0	0		1
GBM	$dr(t) = \beta r(t)dt + \sigma r(t)dZ$	0			1
Brennan-Schwartz	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$				1
CIR VR	$dr(t) = \sigma r^{3/2}(t)dZ$	0	0		$3/2$
CEV	$dr(t) = \beta r(t)dt + \sigma r^\gamma(t)dZ$	0			

These models represent some of the well-known single-factor models in current literature. From the parametric restrictions, it is obvious that the models cannot generally

be written as special cases of one another. That is, although each of the models is nested within (22), they are typically non-nested with respect to each other.

In the CKLS study the continuous-time model, (22), was discretised as follows:

$$r(t+1) - r(t) = \alpha + \beta r(t) + \varepsilon(t+1) \quad (23)$$

$$\text{where } E_t[\varepsilon(t+1)] = 0 \text{ and } E_t[\varepsilon^2(t+1)] = \sigma^2 r^{2\gamma}(t). \quad (24)$$

The parameters of the model were then estimated using the generalised method of moments (GMM) technique. However, as Nowman points out, this discretised model (23) neglects errors introduced as a result of time aggregation. The discretised error arises because equation (22) is only shorthand notation for the stochastic differential equation (SDE),

$$\int_0^t dr(s) = \int_0^t [\alpha + \beta r(s)] ds + \int_0^t \sigma r^\gamma(s) dZ(s) \quad (25)$$

which is the correct representation of the stochastic process.

A more formal approach is to first solve the SDE, (25), for $r(t)$ and then proceed to discretise the solution. This process yields the following discretisation of equation (22),

$$\begin{aligned} r(t) &= \frac{\alpha}{\beta} [e^\beta - 1] + e^\beta r(t-1) + \varepsilon(t) \\ \varepsilon(t) &= \int_{t-1}^t e^{\beta(t-s)} \sigma r^\gamma(s) dZ \end{aligned} \quad (26)$$

where the conditional mean and variance of the error term are given by

$$E_{t-1}[\varepsilon(t)] = 0 \text{ and } E_{t-1}[\varepsilon^2(t)] = \frac{\sigma^2}{2\beta} [e^{2\beta} - 1] r^{2\gamma}(t-1). \quad (27)$$

(See Bergstrom (1984), Nowman (1997) and Appendix B for details of the solution.)

Equation (26) is the exact solution to the PDE (22). Note also that the difference between the discrete time approximation, equations (23) and (24), and the exact solution, equations (26) and (27), lessens as the mean reversion parameter, β , tends to zero.

Section III – Estimation and Data

This section outlines the estimation technique used to estimate the parameters in equation (22) employing the discretisation given by equations (26) and (27). This is the method used by Nowman (1997).

Let the complete set of parameters be defined as $\theta = [\alpha, \beta, \gamma, \sigma^2]$. Under the assumption that the model errors, $\varepsilon(t)$, are conditionally normal, we define the log-likelihood function for (23) or (26) as

$$L(\theta) = -\frac{1}{2} \sum_{t=1}^T \left[\log[2\pi m_t^2] + \frac{\varepsilon^2(t)}{m_t^2} \right] \quad (28)$$

where $m_t^2 = E_{t-1}[\varepsilon^2(t)]$ is the conditional variance and T is the total number of observations¹. The log-likelihood function estimates of the model parameters are then given by

$$\hat{\theta} = \arg \max_{\theta} \{L(\theta)\} \quad (29)$$

where $\hat{\theta}$ is the parameter vector that generates the largest value of $L(\theta)$.

We also estimate the interest rate models with the discrete approximation used in CKLS given by equations (23) and (24).

¹ See Appendix C for the derivation of the likelihood function.

The data used in this study is the three-month South African T-bill yields obtained from *DataStream*. In keeping with the Nowman paper the data are monthly, taken on the 15th of each month, covering the period from January 1957 to December 2005 giving a total of 588 observations.

We will, however, also replicate the study using only data for the period January 1985 to December 2005 since the market for South African bonds and T-bills was relatively illiquid in the years preceding 1985. This can be attributed to the fact that there was virtually no active secondary market trading in government securities in South Africa until 1982 (McLeod, 1990). It is also the beginning of what is recognised as one of the regime changes that occurred within the South African Reserve Bank (SARB) and therefore provides a good starting point for the collection of data (Aron and Muellbauer, 2000).

Figure I shows the T-bill rate and Figure II shows the first differences of the T-bill rate between 1957 and 2005. As can be seen, the volatility of the T-bill rate increases dramatically in the early 1980's. This can be accounted for changes in monetary policy in the SARB in the early 1980's. The prevailing regime between 1957 and the early 1980s was a liquid asset ratio-based system with quantitative controls on interest rates and credit. This was gradually reformed toward a cash reserves-based system, by 1985. Pre-announced, flexible monetary target ranges were used from 1986, with the main policy emphasis on the central bank's discount rate in influencing the cost of overnight collateralized lending and hence market interest rates (Aron and Muellbauer, 2000).

Financial liberalisation from the early 1980s, and a more open capital account in the 1990's, had greatly diminished the usefulness of such targets. They were formally supplemented by a broader set of indicators, including the exchange rate, asset prices, the output gap, the balance of payments, wage settlements, total credit extension, and the fiscal stance (SARB Quarterly Bulletin, October, 1997).

Figure I
The South African Three-Month Treasury Bill Rate
1957 – 2005

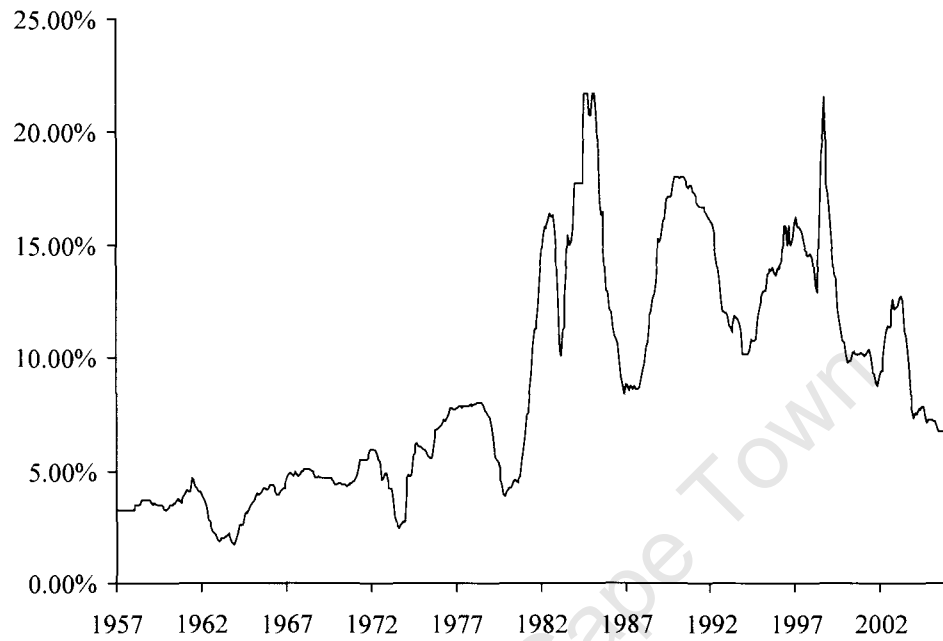
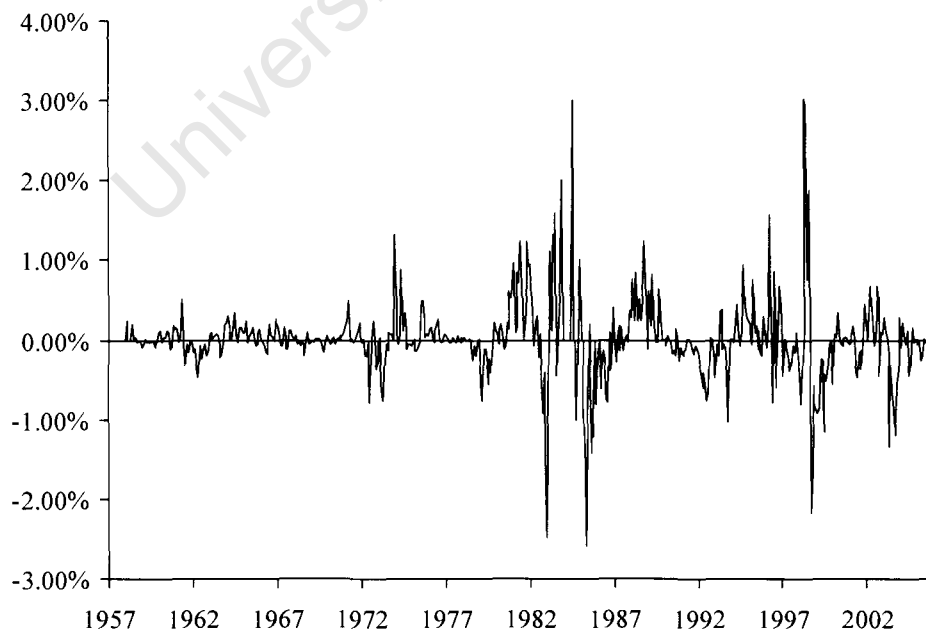


Figure II
The 1st Differences of the South African Three-Month Treasury Bill Rate
1957 – 2005



The descriptive statistics for the period January 1957 to December 2005 can be seen in Table II while the statistics for the period January 1985 to December 2005 can be seen in Table III. The tables display the means, standard deviations and the first six autocorrelations of the three month rate as well as the changes in the three month rate for the two periods. An augmented Dickey-Fuller (ADF) statistic has also been included to test for the presence of a unit root in the data.

The average level of the three month South African T-bill rate for the period January 1957 to December 2005 is 8.87 percent with a standard deviation of 5.03 percent. The autocorrelations for the T-bill rate fall off slowly and those of the first differences are small and neither systematically positive or negative. This indicates the presence of a unit root which is confirmed by the ADF statistic which fails to reject the null hypothesis of a unit root at the 5 percent level of significance.

The average level of the three month South African T-bill rate for the period January 1985 to December 2005 is significantly larger at 12.63 percent with a standard deviation of 3.52 percent. As before the autocorrelations for the T-bill rate fall off slowly and those of the first differences are small and neither systematically positive or negative. This indicates the presence of a unit root which is confirmed by the ADF statistic which again fails to reject the null hypothesis of a unit root at the 5 percent level of significance.

Table II
Summary Statistics
1957 – 2005

Means, standard deviations and autocorrelations of the South African three-month T-bill rate and the first differences are computed for the series January 1957 to December 2005. The variable $r(t)$ denotes the three-month T-bill rate and $\Delta r(t)$ denotes the monthly change. ρ_j denotes the autocorrelation coefficient of order j . T represents the number of observations used. ADF denotes the Augmented Dickey-Fuller unit root statistic with a 5 percent critical value of -2.860.

Variable	T	Mean	Standard Deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ADF
$r(t)$	588	0.0887	0.0503	0.994	0.984	0.972	0.958	0.944	0.928	-1.387
$\Delta r(t)$	587	0.0061	0.0197	0.483	0.248	0.156	0.084	0.100	0.085	-14.270

Table III
Summary Statistics
1957 – 2005

Means, standard deviations and autocorrelations of the South African three-month T-bill rate and the first differences are computed for the series January 1985 to December 2005. The variable $r(t)$ denotes the three-month T-bill rate and $\Delta r(t)$ denotes the monthly change. ρ_j denotes the autocorrelation coefficient of order j . T represents the number of observations used. ADF denotes the Augmented Dickey-Fuller unit root statistic with a 5 percent critical value of -2.880.

Variable	T	Mean	Standard Deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ADF
$r(t)$	252	0.1263	0.0352	0.968	0.924	0.872	0.817	0.766	0.720	-2.078
$\Delta r(t)$	251	0.0005	0.0056	0.483	0.271	0.191	0.030	0.082	0.067	-9.298

Section IV – Results

In this section we present the Gaussian estimation results from the unrestricted model and the eight nested term structure models obtained after imposing the necessary restrictions on the general model. We compare the explanatory power of these different models compared to the unrestricted model by comparing the maximised Gaussian likelihood function values and performing likelihood ratio tests.

A. Results for the Period 1957 – 2005

In Table IV we present the Gaussian coefficient estimates, their standard deviations, maximised log likelihoods for the unrestricted and eight nested models, and the likelihood ratio tests comparing the nested models with the unrestricted model.

A comparison of the Gaussian estimates confirms Nowman's finding that the asymptotic bias resulting from the CKLS approximation is very small. The estimates are almost identical under both the CKLS approximation and the discrete approximation proposed by Nowman.

We will now focus on the results generated using the discrete approximation proposed by Nowman. Based on the χ^2 likelihood ratio test under the null hypothesis that the nested model restrictions are valid we can reject the Merton, Vasicek, CIR SR, Dothan, GBM, Brennan-Schwartz and CIR VR models. We only fail to reject the CEV model which also performs best when comparing its maximised Gaussian likelihood value with that of the unrestricted model.

Based on the maximised Gaussian likelihood values compared with that of the unrestricted model the CEV model performs best followed by the CIR SR model. Since the unrestricted, CIR SR and CEV models all include a gamma coefficient greater than zero we can conclude that there is strong evidence that the conditional volatility is dependent on the level of the interest rate. The unrestricted and CEV models estimate gamma at 0.4127 and 0.4120 respectively and both of these estimates are significant at the 1 percent level.

There is no evidence of a linear trend: in all the models, estimates of α are close to zero, and are all insignificant with the exception of the Brennan-Schwartz model which has a significant positive result. This is extremely small and, thus we conclude that there is no significant linear trend. There is however some evidence of mean reversion since the unrestricted, Vasicek, CIR SR and Brennan-Schwartz models all have negative estimates of β . However, none of these estimates are significant and thus while there may be a mean reversion effect we can conclude that it is extremely small.

Table IV
Gaussian Estimates of Continuous-time Models of the Short-Term Interest Rate
1957 – 2005

Gaussian estimates of alternative one factor models of the short-term interest rate $r(t)$ (three month South African T-bill rate) from January 1957 to December 2005 (587 observations). The models are

Unrestricted	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^\gamma(t)dZ$
Merton	$dr(t) = \alpha dt + \sigma dZ$
Vasicek	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$
CIR SR	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$
Dothan	$dr(t) = \sigma r(t)dZ$
GBM	$dr(t) = \beta r(t)dt + \sigma r(t)dZ$
Brennan-Schwartz	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$
CIR VR	$dr(t) = \sigma r^{3/2}(t)dZ$
CEV	$dr(t) = \beta r(t)dt + \sigma r^\gamma(t)dZ$

Gaussian estimates with their standard deviations in parentheses are presented for each model. Estimates marked with * are significant at the 5 percent level. Likelihood ratio tests evaluate restrictions imposed by different models against the unrestricted model. The χ^2 test statistics are reported with p -values in parentheses and their associated degrees of freedom (d.f.). The Gaussian estimates are obtained from the following system of equations

$$r(t) = \frac{\alpha}{\beta}[e^\beta - 1] + e^\beta r(t-1) + \varepsilon(t)$$

$$E_{t-1}[\varepsilon(t)\varepsilon(s)] = 0, \quad t \neq s$$

$$E_{t-1}[\varepsilon^2(t)] = \frac{\sigma^2}{2\beta}[e^{2\beta} - 1]r^{2\gamma}(t-1)$$

The CKLS Gaussian estimates are obtained from the following system

$$r(t+1) - r(t) = \alpha + \beta r(t) + \varepsilon(t+1)$$

$$E_t[\varepsilon(t+1)] = 0$$

$$E_t[\varepsilon^2(t+1)] = \sigma^2 r^{2\gamma}(t)$$

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	d.f.
Unrestricted	0.000318 (0.00021)	-0.00273 (0.00362)	0.031088* (0.00345)	0.412669* (0.02062)	2464.3837		
CKLS	0.000318 (0.00021)	-0.002726 (0.00361)	0.031003* (0.00344)	0.412669* (0.02061)	2464.3839		
Merton	0.000061 (0.00019)	0.0	0.004775* (0.00014)	0.0	2304.2874	320.1926 (< 0.0001)	2
CKLS	0.000061 (0.00019)	0.0	0.004775* (0.00014)	0.0	2304.2874	320.1926 (< 0.0001)	2
Vasicek	0.000545 (0.00040)	-0.005455 (0.00394)	0.004793* (0.00014)	0.0	2305.251	318.2654 (< 0.0001)	1
CKLS	0.000544 (0.00399)	-0.005440 (0.00392)	0.004766* (0.00014)	0.0	2305.251	318.2654 (< 0.0001)	1
CIR SR	0.000316 (0.00019)	-0.002672 (0.00381)	0.049698* (0.00146)	0.5	2455.239	18.2894 (< 0.0001)	1
CKLS	0.000315 (0.00019)	-0.002668 (0.00379)	0.049565* (0.00145)	0.5	2455.239	18.2894 (< 0.0001)	1
Dothan	0.0	0.0	1.229314* (0.03507)	1.0	2094.9664	738.8346 (< 0.0001)	3
CKLS	0.0	0.0	1.229314* (0.03507)	1.0	2094.9664	738.8346 (< 0.0001)	3
GBM	0.0	0.006902* (0.00260)	1.213653* (0.03556)	1.0	2098.4368	731.8938 (< 0.0001)	2
CKLS	0.0	0.006926* (0.00262)	1.222068* (0.03567)	1.0	2098.4368	731.8938 (< 0.0001)	2
Brennan-Schwartz	0.000449* (0.00020)	-0.006306 (0.00646)	1.22452* (0.03660)	1.0	2100.9578	726.8518 (< 0.0001)	1
CKLS	0.000448* (0.00020)	-0.006286 (0.00642)	1.21683* (0.03551)	1.0	2100.9578	726.8518 (< 0.0001)	1
CIR VR	0.0	0.0	46.4279* (1.35502)	1.5	1487.8452	1953.077 (< 0.0001)	3
CKLS	0.0	0.0	46.4279* (1.35502)	1.5	1487.8452	1953.077 (< 0.0001)	3
CEV	0.0	0.001787 (0.00199)	0.030902* (0.00342)	0.412014* (0.02060)	2463.261	2.2454 (0.13401)	1
CKLS	0.0	0.001788 (0.00200)	0.030957* (0.00343)	0.412013* (0.02060)	2463.261	2.2454 (0.13401)	1

B. Results for the Period 1985 – 2005

In Table V we present the results for the estimation using data from January 1985 to December 2005. This shorter period is interesting since there is evidence of a structural break in the data series in the mid-1980's; the market for South African treasury bills was very illiquid in the years preceding 1980. Once again we observe that the asymptotic bias resulting from the CKLS approximation is extremely small and that the estimates are

almost identical for both the CKLS approximation and the discrete approximation used by Nowman.

We now concentrate on the Gaussian estimation results generated using the discrete approximation of Nowman (1997). Based on the χ^2 likelihood ratio tests under the null hypothesis that the nested model restrictions imposed are valid we can reject the Merton, Vasicek, CIR SR, Dothan, GBM, Brennan-Schwartz and CIR VR models. Once again the only model that we fail to reject is the CEV model which also performs best when comparing the maximised Gaussian likelihood values against that of the unrestricted model. This is evidence that the underlying process describing the South African short rate did not change in the 1980's.

Based on the maximised Gaussian likelihood values compared with that of the unrestricted model the CEV model once again performs best followed by the Brennan-Schwartz, GBM and CIR SR model. Since all these models include a γ coefficient greater than zero we can conclude that there is strong evidence that the conditional volatility is dependent on the level of the interest rate. The unrestricted and CEV models estimate γ at 0.7548 and 0.7592 respectively and both of the estimates are significant at the 1 percent level. This is somewhat higher than the estimate we yield when using the entire data series. This provides evidence that, while the underlying process driving the interest rate may not have changed in the period after 1980, the effect of the level of the interest rate on the conditional volatility may have become more pronounced. A possible explanation for this might be that more heavily traded markets are subject to higher levels of volatility.

There is small evidence of mean reversion since all the models have negative estimates of the beta coefficient. However, these estimates are only slightly negative and insignificant and thus we can conclude that while there is some evidence of mean reversion in the data, the effect is small and perhaps surprisingly, insignificant. There appears to be no evidence of a linear trend in the data since all the models have estimates of α that are very close to zero and all are insignificant.

Table V
Gaussian Estimates of Continuous-time Models of the Short-Term Interest Rate
1985 – 2005

Gaussian estimates of alternative one factor models of the short-term interest rate $r(t)$ (three month South African T-bill rate) from January 1957 to December 2005 (587 observations). The models are

Unrestricted	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^\gamma(t)dZ$
Merton	$dr(t) = \alpha dt + \sigma dZ$
Vasicek	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$
CIR SR	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$
Dothan	$dr(t) = \sigma r(t)dZ$
GBM	$dr(t) = \beta r(t)dt + \sigma r(t)dZ$
Brennan-Schwartz	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$
CIR VR	$dr(t) = \sigma r^{3/2}(t)dZ$
CEV	$dr(t) = \beta r(t)dt + \sigma r^\gamma(t)dZ$

Gaussian estimates with their standard deviations in parentheses are presented for each model. Estimates marked with * are significant at the 5 percent level. Likelihood ratio tests evaluate restrictions imposed by different models against the unrestricted model. The χ^2 test statistics are reported with p-values in parentheses and their associated degrees of freedom (d.f.). The Gaussian estimates are obtained from the following system of equations

$$r(t) = \frac{\alpha}{\beta} [e^\beta - 1] + e^\beta r(t-1) + \varepsilon(t)$$

$$E_{t-1}[\varepsilon(t)\varepsilon(s)] = 0, \quad t \neq s \qquad E_{t-1}[\varepsilon^2(t)] = \frac{\sigma^2}{2\beta} [e^{2\beta} - 1] r^{2\gamma}(t-1)$$

The CKLS Gaussian estimates are obtained from the following system

$$r(t+1) - r(t) = \alpha + \beta r(t) + \varepsilon(t+1)$$

$$E_t[\varepsilon(t+1)] = 0 \qquad E_t[\varepsilon^2(t+1)] = \sigma^2 r^{2\gamma}(t)$$

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	d.f.
Unrestricted	0.000515 (0.00093)	-0.007965 (0.00906)	0.117098* (0.04394)	0.754814* (0.08865)	982.24387		
CKLS	0.000513 (0.00093)	-0.007933 (0.00899)	0.116163* (0.04368)	0.754798* (0.08864)	982.24387		
Merton	-0.000595 (0.00035)	0.0	0.005619* (0.00025)	0.0	944.40245	75.68284 (< 0.0001)	2
CKLS	-0.000595 (0.00035)	0.0	0.005619* (0.00025)	0.0	944.40245	75.68284 (< 0.0001)	2
Vasicek	0.002083 (0.00135)	-0.02120* (0.01027)	0.005691* (0.00026)	0.0	946.55938	71.36898 (< 0.0001)	1
CKLS	0.002061 (0.00132)	-0.02098* (0.01005)	0.005572* (0.00025)	0.0	946.55938	71.36898 (< 0.0001)	1
CIR SR	0.000867 (0.00103)	-0.011245 (0.00919)	0.040828* (0.00186)	0.5	978.11091	8.26592 (0.00404)	1
CKLS	0.000862 (0.00102)	-0.011182 (0.00908)	0.040373* (0.00180)	0.5	978.11091	8.26592 (0.00404)	1
Dothan	0.0	0.0	0.332301* (0.01483)	1.0	977.68472	9.1183 (0.02776)	3
CKLS	0.0	0.0	0.332301* (0.01483)	1.0	977.68472	9.1183 (0.02776)	3
GBM	0.0	-0.002785 (0.00235)	0.332298* (0.01485)	1.0	978.38551	7.71672 (0.02110)	2
CKLS	0.0	-0.002781 (0.00235)	0.331375* (0.01479)	1.0	978.38551	7.71672 (0.02110)	2
Brennan-Schwartz	0.000290 (0.00087)	-0.005707 (0.00904)	0.333195* (0.01517)	1.0	978.44173	7.60428 (0.00582)	1
CKLS	0.000289 (0.00086)	-0.005691 (0.00899)	0.331300* (0.14787)	1.0	978.44173	7.60428 (0.00582)	1
CIR VR	0.0	0.0	3.080752* (0.13750)	1.5	947.39002	69.7077 (< 0.0001)	3
CKLS	0.0	0.0	3.080752* (0.13750)	1.5	947.39002	69.7077 (< 0.0001)	3
CEV	0.0	-0.003160 (0.00247)	0.118787* (0.04439)	0.759207* (0.08811)	982.09125	0.30524 (0.58062)	1
CKLS	0.0	-0.003155 (0.00246)	0.118411* (0.04426)	0.759205* (0.00811)	982.09125	0.30524 (0.58062)	1

Section V – Conclusion

In this paper we present an application of the Gaussian estimation techniques developed by Bergstrom (1983, 1984, 1985, 1986, 1990) for the estimation of continuous-time dynamic models. We follow the methodology used by Nowman (1997) and estimate a range of single-factor continuous-time models of the short-term interest rate using data on the three month South African T-bill rate. We adopt the approach used by CKLS whereby eight well-known models of the short-term interest rate can be nested within a general

stochastic differential equation. We obtain estimates using the discrete model proposed by Nowman (1997) and compare them with the estimates obtained using the discrete approximation of CKLS. We find that the asymptotic bias that results from using the discrete approximation of CKLS is very small.

Based on the Gaussian estimates obtained using the discrete model proposed by Nowman (1997) we find evidence that the volatility of the South African short-term interest rate is somewhat sensitive to the level of the interest rate. This effect appears to be greater in the period between 1985 and 2005 (gamma is approximately 0.75) than over the entire period between 1957 and 2005 (gamma is approximately 0.41). This confirms our suspicion that there is a structural break in the data series that occurs in the mid 1980's. This break can be explained by market liberalisation that occurred in the 1980's and the very thin market for South African treasury bills that existed before that. We find that for both series, 1957 to 2005 and 1985 to 2005, there is strong evidence that the CEV model best describes the process that the South African short-term interest rate follows.

Appendix A - Change-of-Numeraire Technique

(Brigo and Mercurio, 2001, p.27)

The pricing formula gives the unique no-arbitrage price of an attainable contingent claim H in terms of the expectation of the claim payoff under selected martingale measure Q . However, this measure is not necessarily the most natural and convenient measure for pricing the claim H .

Definition A 1.1 (Brigo and Mercurio, 2001, p.27, Definition 2.2.1)

Any non-dividend paying asset in the model of which the price is always strictly positive can be taken as numeraire.

In general, a numeraire Z is identifiable with a self-financing strategy ϕ in that $Z_t = V_t(\phi)$ for each t . Intuitively, a numeraire is a reference asset that is chosen so as to normalize all other asset prices with respect to it. Thus, choosing a numeraire Z implies that relative prices S^k/Z $k = 0, 1, \dots, n$ are considered instead of the asset prices themselves.

The following proposition provides a fundamental tool for the pricing of derivatives and is the natural generalisation of Definition 1.2.2 to any numeraire.

Proposition A 1.1 (Brigo and Mercurio, 2001, p.27, Proposition 2.2.1)

Let N be a numeraire, and Q_0 the equivalent martingale measure for the numeraire B .

Then Q^N defined by

$$Q^N(A) := \frac{B(0,0)}{N(0)} \int_A \frac{N(T)}{B(0,T)} dQ_0 = \frac{1}{N(0)} \int_A N(T) D(0,T) dQ_0 \quad \forall A \in F(T)$$

is an equivalent martingale measure for Q .

An equivalent way of expressing (2.3.1) is to say

$$Q^N \sim Q_0 \text{ with Radon-Nikodym derivative } \frac{Q^N}{Q_0} \Big|_{F_t} = \frac{B(0,0)N(t)}{N(0)B(0,t)}$$

Appendix B – Solving the Stochastic Differential Equation

(McManus and Watt, 1999, p.27)

Consider the stochastic differential equation

$$dr(t) = [\alpha + \beta r(t)]dt + \phi(r, t)dZ. \quad (A1)$$

The above equation can be solved by first introducing a variable $Y(t) = \alpha + \beta r(t)$ and then using Itô's Lemma to notice that the following equation must hold,

$$d[e^{-\beta t} Y(t)] = \beta e^{-\beta t} \phi(r, t)dZ \quad (A2)$$

Equation (A1) is simply short hand for

$$[e^{-\beta s} Y(s)]' \Big|_0^t = \int_0^t \beta e^{-\beta s} \phi(r, s)dZ. \quad (A3)$$

Simplifying equation (A2) yields the general solution

$$r(t) = -\frac{\alpha}{\beta} + \frac{1}{\beta} e^{-\beta t} [\alpha + \beta r(0)] + e^{\beta t} \int_0^t \beta e^{-\beta s} \phi(r, s)dZ. \quad (A4)$$

The solution can be written in iterative form, namely,

$$\begin{aligned} r(t) &= \frac{\alpha}{\beta} (e^{\beta} - 1) + e^{\beta} r(t-1) + \varepsilon(t) \\ \varepsilon(t) &= \int_{t-1}^t e^{\beta(t-s)} \phi(r, s)dZ \end{aligned} \quad (A5)$$

Thus equation (A4) is the correct discretisation model for the stochastic differential equation (25). Equation (A4) takes care of the aggregation over time issues. Note that the conditional mean and variance of the error term are given by

$$E_{t-1}[\varepsilon(t)] = 0 \quad (\text{A6})$$

$$\begin{aligned} E_{t-1}[\varepsilon^2(t)] &= E_{t-1}\left[\left\{\int_{t-1}^t e^{\beta(t-s)}\phi(r,s)dZ\right\}^2\right] \\ &= E_{t-1}\left[\int_{t-1}^t e^{2\beta(t-s)}\phi^2(r,s)ds\right] \quad \text{by Ito Isometry} \\ &= \int_{t-1}^t e^{2\beta(t-s)}E_{t-1}[\phi^2(r,s)]ds \end{aligned} \quad (\text{A7})$$

To proceed further, let $\phi(r,s) = \sigma r(t)^\gamma$ and approximate $E_{t-1}[\varepsilon^2(t)]$ by $\sigma^2 r(t-1)^{2\gamma}$. The condition variance of the error term can then be approximated by

$$E_{t-1}[\varepsilon^2(t)] = \frac{\sigma^2}{2\beta} (e^{2\beta} - 1) r(t-1)^{2\gamma}. \quad (\text{A8})$$

Appendix C – Derivation of the likelihood function

Suppose X_1, X_2, \dots, X_T are independent, identically distributed and are conditionally normally distributed with mean μ and conditional variance σ^2 . Then the joint probability distribution of these variables is

$$\begin{aligned} f(x_1, \dots, x_T, \mu, \phi) &= (\phi\sqrt{2\pi})^{-T} \prod_{i=1}^T \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu}{\phi}\right)^2\right] \\ &= (\phi\sqrt{2\pi})^{-T} \exp\left[-\frac{1}{2}\left(\frac{\sum_{i=1}^T (x_i - \mu)^2}{\phi^2}\right)\right] \end{aligned} \quad (\text{A9})$$

This is the likelihood function. We now define $L(\theta)$ as the log of the likelihood function.

$$\begin{aligned} L(\theta) &= \log\left[(\phi\sqrt{2\pi})^{-T} \exp\left[-\frac{1}{2}\left(\frac{\sum_{i=1}^T (x_i - \mu)^2}{\phi^2}\right)\right]\right] \\ &= \log\left[(\phi\sqrt{2\pi})^{-T}\right] + \log\left[\exp\left[-\frac{1}{2}\left(\frac{\sum_{i=1}^T (x_i - \mu)^2}{\phi^2}\right)\right]\right] \\ &= -\frac{1}{2}T \log[\phi^2 2\pi] - \frac{1}{2}\left(\frac{\sum_{i=1}^T (x_i - \mu)^2}{\phi^2}\right) \\ &= -\frac{1}{2} \sum_{i=1}^T \left\{ \log[\phi^2 2\pi] + \left(\frac{x_i - \mu}{\phi}\right)^2 \right\} \end{aligned} \quad (\text{A10})$$

Now in our case

$$x_t = \varepsilon(t), \mu = E_{t-1}[\varepsilon(t)] = 0 \text{ and } \phi^2 = E_{t-1}[\varepsilon^2(t)] = \frac{\sigma^2}{2\beta} [e^{2\beta} - 1] r^{2\gamma} (t-1) = m_t^2.$$

Substituting these into (A10) yields the following likelihood function which is the same as (28)

$$L(\theta) = -\frac{1}{2} \sum_{t=1}^T \log[2\pi m_t^2] + \frac{\varepsilon^2(t)}{m_t^2} \quad (\text{A11})$$

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